# Functions

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If we divide two of these numbers, we find that for instance

$$\frac{4}{5} = 0,8 \notin \mathbb{Z}.$$

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- So people filled the gap between any two rational numbers and stated so the real numbers denoted by

$$\mathbb{R} = \mathbb{Q} \cup \{\cdots, \sqrt{2}, \cdots, e, \cdots, \pi, \cdots\}.$$

**Defintion (Functions).**

Let $A$ and $B$ be non-empty sets. If every element $x \in A$ is assigned **exactly** with one element $y = f(x) \in B$, then $f$ is called a function or a mapping from $A$ to $B$.

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**Examples.**

- Let $A := \{x\}$ and $B := \{y_1, y_2\}$. Then $f : A \to B$ with

$$f(x) = y_1 \text{ and } f(x) = y_2$$

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- The mapping

  $$f : A \to A, \;\; x \mapsto f(x) = x$$

  for all $x \in A$ is called the **identity** on $A$.

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Let $f : A \to B$ be a function. The set

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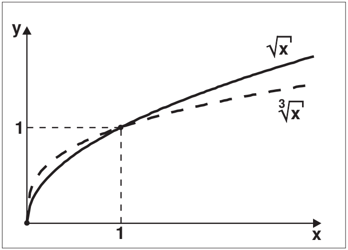
Note that the image is always a subset of the codomain.

**Example.**

Consider the *n-th root function* $f : [0, \infty) \to \mathbb{R}$ given by

$$f(x) := \sqrt[n]{x} \quad \text{for all} \quad x \geq 0, \quad n \in \mathbb{N} \quad \text{fixed.}$$

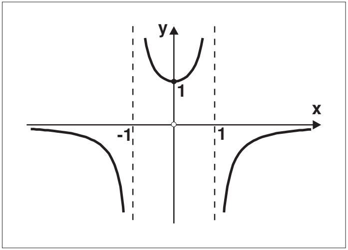



$D_f = [0, +\infty) = f(D_f)$

**Example.**

Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$\boxed{f(x) := \frac{1}{1-x^2}.}$$

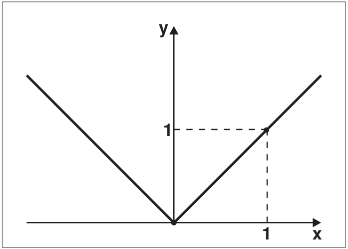



$D_f = \mathbb{R} \setminus \{-1, 1\}$, $f(D_f) = \mathbb{R} \setminus [0, 1)$

**Example.**

Consider the absolut value function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\boxed{f(x) := |x| \quad \text{for all} \;\; x \in \mathbb{R}.}$$




$D_f = \mathbb{R}, \;\; f(D_f) = [0, +\infty).$

In general we have

**Definition.**

For $x \in \mathbb{R}$ the expression

$$|x| := \begin{cases} x & : \quad x \geq 0, \\ -x & : \quad x < 0, \end{cases}$$

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Usefull inequality

$$|x| \leq c \iff -c \leq x \leq c.$$

**Example.**

Find all $x \in \mathbb{R}$ which solve the inequality

$$|x - 5| < 4.$$

Solution:

$$|x - 5| < 4 \quad \Leftrightarrow \quad -4 < x - 5 < 4 \quad \Longleftrightarrow \quad 1 < x < 9.$$

In general we have

**Remark.**

The real solutions of the equation $x^n = a$ are

$$a \geq 0 : \quad x = \begin{cases} \pm\sqrt[n]{a} & : \quad n = 2k \qquad\qquad (n \text{ even}), \\ \sqrt[n]{a} & : \quad n = 2k+1 \qquad (n \text{ odd}). \end{cases}$$

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We can write

$$a^{1/n} = \sqrt[n]{a}.$$

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One has to be careful when using the notation $a^{1/n} = \sqrt[n]{a}$ because, for example,

$$\sqrt[3]{-8} = -2,$$

but

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**Note.**

The number under any root has to be positive!

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Solution: The roots are defined for any $x \in \mathbb{R}$. Thus we get

$$(2\sqrt[3]{x-1})^3 < (\sqrt[3]{x+13})^3 \;\Leftrightarrow\; 8(x-1) < x+13 \;\Leftrightarrow\; x < 3.$$

Together we have

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A test confirms our solution. Never forget the test, especially if you raise even powers! Why?

**Definition.**

If $f$ and $g$ are two functions with common domain $A$, we can define

- $f \pm g : A \to \mathbb{R}, \quad x \mapsto f(x) \pm g(x),$
- $f \cdot g : A \to \mathbb{R}, \quad x \mapsto f(x)g(x),$
- $\frac{f}{g} : A \to \mathbb{R}, \quad x \mapsto \frac{f(x)}{g(x)}$ for all $x$ with $g(x) \neq 0$.

**Example.**

Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $b_0, b_1, \ldots, b_m \in \mathbb{R}$ be fixed numbers. $P_n$ and $Q_m$ are polynomials of degree $n$ and $m$, respectively.
The rational function $h : \mathbb{R} \to \mathbb{R}$ given by

$$\boxed{h(x) := \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}.}$$

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The domain of the function defined by the previous expression is

$$D_h := \{x \in \mathbb{R} \, : \, Q_m(x) \neq 0\}.$$

## Some properties of functions.

**Definition.**

- Let $f : A \to B$ with $A, B \subset \mathbb{R}$. The function $f$ increases monotonically, if

$$x < x' \implies f(x) \leq f(x').$$

- $f$ increases strictly monotonically, if

$$x < x' \implies f(x) < f(x').$$

- $f$ decreases monotonically or decreases strictly monotonically, if

$$x < x' \implies f(x) \geq f(x') \text{ or } f(x) > f(x'), \text{ respectively.}$$

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Note: Monotonicity depends on the domain.

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Note: Boundedness depends on the domain of the function $f$.

# Inverse function

**Definition.**

Let $f : A \to B$ be a function. Then $f$ is said to be invertible, if there exists for every $y \in B$ exactly one $x \in A$ with $y = f(x)$. The function

$$f^{-1} : B \to A, \quad y \mapsto f^{-1}(y), \quad f^{-1}(y) = x$$

is called inverse function of $f$.

## Examples

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We check the first example:

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We switch the variables to get

$$f^{-1}(x) = \frac{1}{3}(x - 2).$$

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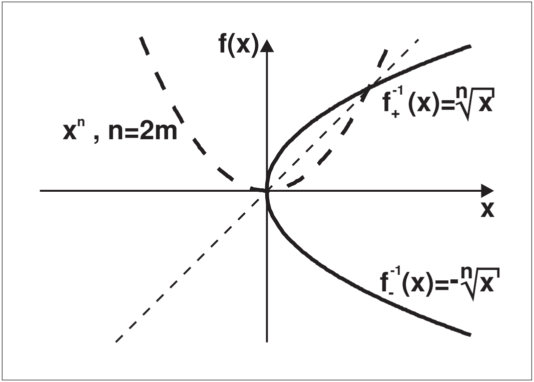


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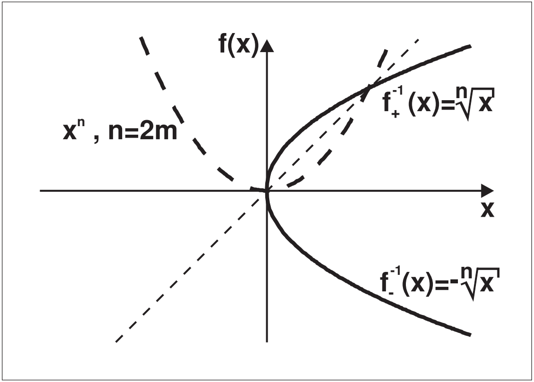
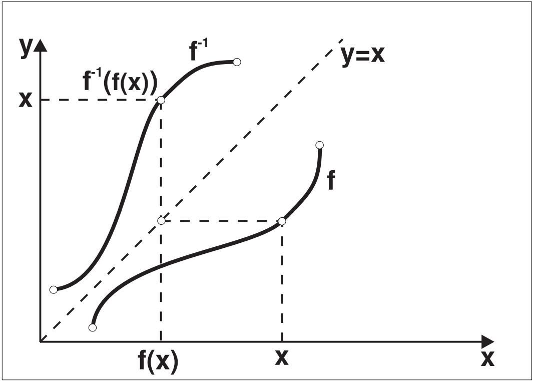



**Inverses of** $\mathbf{f(x) := x^{2m}, m \in \mathbb{N}}$

What about odd powers?

**Remark.**

It is easy to obtain the graph of the inverse function $f^{-1}$ from the graph of $f$. By interchanging the variables $x$ and $y$, as we did in the example above, we reflect the graph at the bisector $y = x$.




**Reflection across** $y = x$

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If this is not the case over the whole domain, we determine a partial inverse by restricting the domain.

**Definition.**

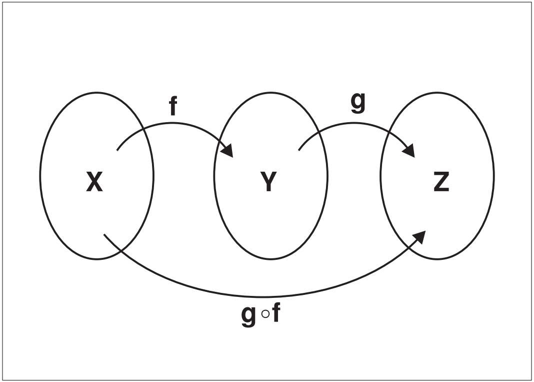
Let $X, Y, Z$ be nonempty sets with $x \in X$, $y \in Y$ und $z \in Z$. We consider the functions

$$f : X \to Y \text{ and } g : Y \to Z.$$

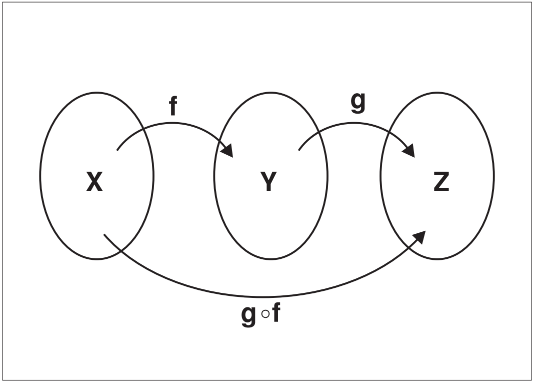
Then the mapping

$$(g \circ f) : X \to Z \text{ given by } (g \circ f)(x) := g(f(x)) = g(y) = z$$

is called the composition of $f$ and $g$.




**Composition of *f* and *g***




**Composition of *f* and *g***

We have the relation

$$\boxed{f\big(f^{-1}(y)\big) = y \quad \text{and} \quad f^{-1}\big(f(x)\big) = x.}$$

**Example.**

Let

- $f : \mathbb{R} \to [1, \infty)$ given by $f(x) = x^2 + 1$,
- $g : [1, \infty) \to [1, \infty)$ given by $g(y) = \sqrt{y}$.

Then we compose $f$ and $g$ as

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- What about $(f \circ g)$?

# Functions in $n$ variables

**Definition.**

Functions in $n$ variables are of the form

$$f : \mathbb{R}^n \to \mathbb{R}, \quad u = f(x_1, x_2, \ldots, x_n)$$

with domain $D_f \subset \mathbb{R}^n$ and codomain $u \in \mathbb{R}$.

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**Remark.**

We call

$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

an ordered $n$-tuple.

Only in the case of $D_f \subset \mathbb{R}^2$ we are able to represent

$$u = f(x, y), \quad (x, y) \in D_f,$$

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**Definition.**

The graph

$$G_f := \{(x, y, u) \in \mathbb{R}^3 \,:\, u = f(x, y),\ \ (x, y) \in D_f\}$$

of a function $f$ in two variables is called surface in $\mathbb{R}^3$.

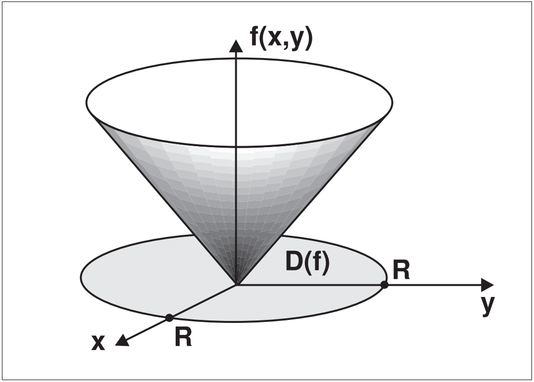
## Example: Surface of a cone

The domain given by

$$D_f := \{(x, y) \in \mathbb{R}^2 \;:\; 0 \leq x^2 + y^2 \leq R^2\}$$

is the circle with radius $R$. The cone is

$$\boxed{u = f(x, y) := \sqrt{x^2 + y^2}}\,.$$



**Example: Surface of a hemisphere**

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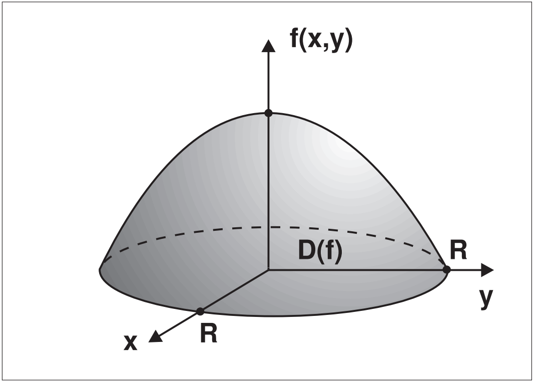


**Definition.**

Let $h \in \mathbb{R}$ be a given number. For $u = f(x, y)$ with $(x, y) \in D_f \subset \mathbb{R}^2$ we call

$$\Gamma_h := \{(x, y) \in D_f \, : \, f(x, y) = h\}$$

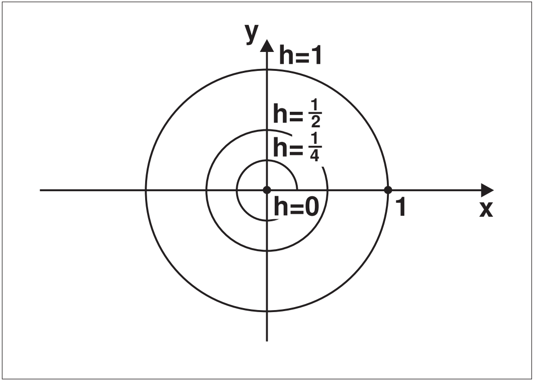
the level curves of $f$.

## Level lines for the surface of the cone

The level lines of the cone are the lines

$$\sqrt{x^2 + y^2} = h = const, \quad h \geq 0,$$

which are concentric circles with radius $h$ around the origin 0.



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In the case of $n = 3$ however we can state

**Definition.**

Let $h \in \mathbb{R}$ be a given number. For $u = f(x, y, z)$ with $(x, y, z) \in D_f \subset \mathbb{R}^3$ we call

$$F_h := \{(x, y, z) \in D_f \,:\, f(x, y, z) = h\}, \quad h \in \mathbb{R},$$

the level surfaces of $f$.

**Example**

The level surfaces of the function

$$f(x, y, z) := x^2 + y^2 - 2z$$

are the paraboloids

$$2z + h = x^2 + y^2, \;\; h = \text{const},$$

which rotate around the $z$–axis.